

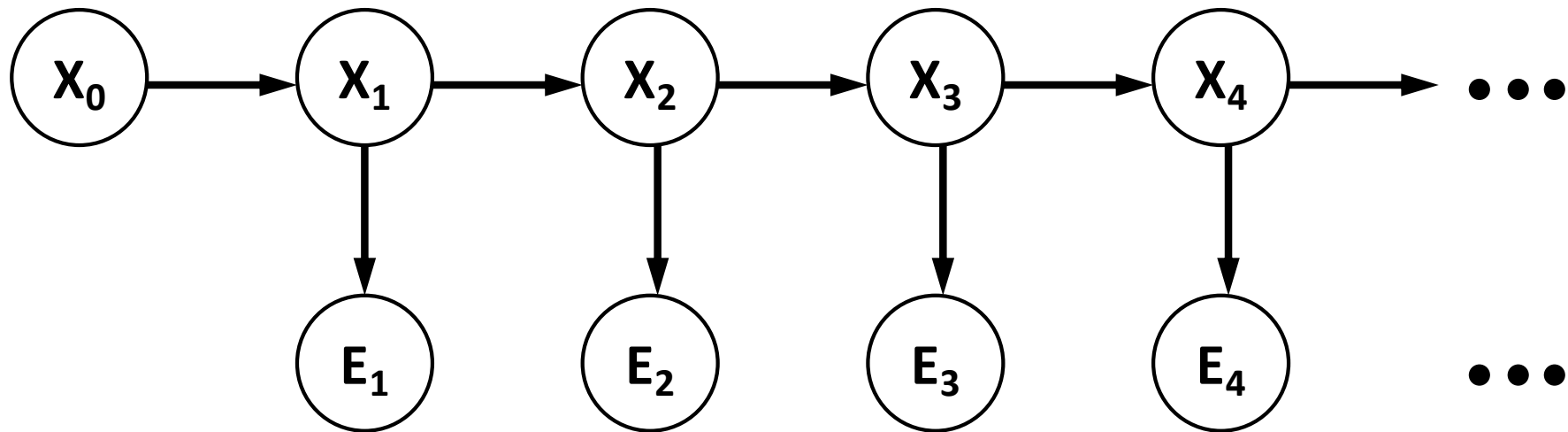
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Introduction to Intelligent Systems

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KALMAN FILTERING

Kalman Filtering vs. Hidden Markov Model

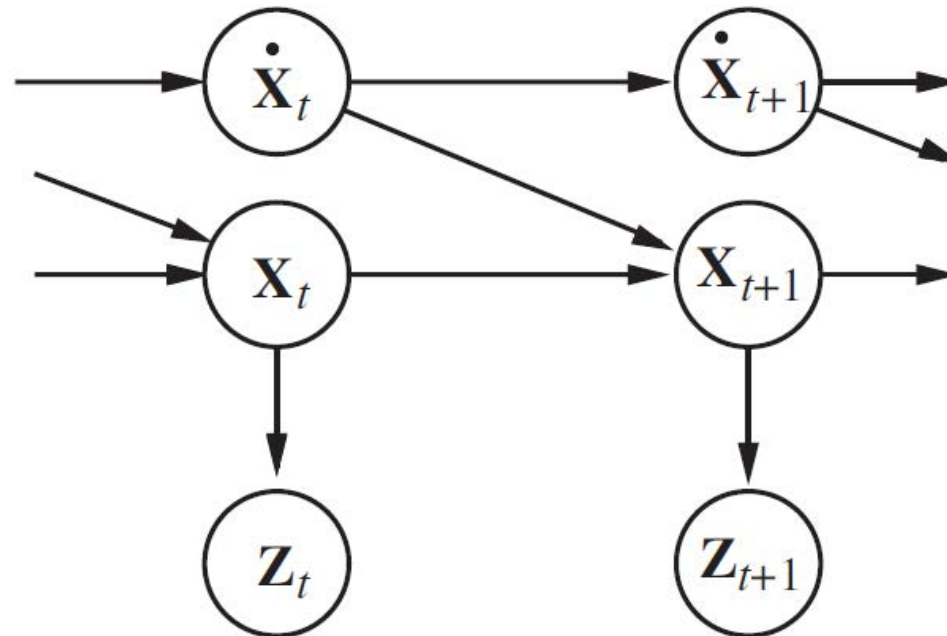


- **Hidden Markov model:** Discrete state space, $X_t \in \{1, \dots, S\}$.
- **Kalman filtering:** Continuous state space, $X_t \in \mathbb{R}^d$. Linear-Gauss Model.

Kalman Filtering

- **Kalman filtering:** Continuous state space, $X_t \in \mathbb{R}^d$. Linear-Gauss Model.
 - Linear Gaussian transition model: $X_{t+\Delta} = X_t + \dot{X}_t\Delta + W_t$, where Δ is the time interval between observations and $W_t \sim \mathcal{N}(0, \sigma^2)$.

$$P(X_{t+\Delta} = x_{t+\Delta} | X_t = x_t, \dot{X}_t = \dot{x}_t) = \mathcal{N}(x_{t+\Delta}; x_t + \dot{x}_t\Delta, \sigma^2)$$



Properties of the Gaussian Distribution

1. If the current distribution $\mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t})$ is Gaussian and the transition model $\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t)$ is linear Gaussian, then the one-step predicted distribution given by

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) = \int_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t}) d\mathbf{x}_t \quad (15.17)$$

is also a Gaussian distribution.

2. If the prediction $\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t})$ is Gaussian and the sensor model $\mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1})$ is linear Gaussian, then, after conditioning on the new evidence, the updated distribution

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \quad (15.18)$$

is also a Gaussian distribution.

- *Forward* operator for Kalman filtering: $\mathbf{f}_{1:t}$ contains μ_t (mean) and Σ_t (covariance matrix).
- Initialization: $\mathbf{f}_{1:0} = P(X_0) = \mathcal{N}(\mu_0, \Sigma_0)$.

Simple One-Dimensional Example

- Prior distribution: $P(X_0) = \mathcal{N}(\mu_0, \sigma_0^2)$

$$P(x_0) = \alpha e^{-\frac{1}{2} \left(\frac{(x_0 - \mu_0)^2}{\sigma_0^2} \right)}$$

- Transition model: $X_{t+1} = X_t + W_t$, where $W_t \sim \mathcal{N}(0, \sigma_x^2)$

$$P(x_{t+1} | x_t) = \alpha e^{-\frac{1}{2} \left(\frac{(x_{t+1} - x_t)^2}{\sigma_x^2} \right)}$$

- Sensor model: $Z_t = X_t + V_t$, where $V_t \sim \mathcal{N}(0, \sigma_z^2)$

$$P(z_t | x_t) = \alpha e^{-\frac{1}{2} \left(\frac{(z_t - x_t)^2}{\sigma_z^2} \right)}$$

One Step Prediction

$$\begin{aligned} P(x_1) &= \int_{-\infty}^{\infty} P(x_1 | x_0) P(x_0) dx_0 = \alpha \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(\frac{(x_1 - x_0)^2}{\sigma_x^2} \right)} e^{-\frac{1}{2} \left(\frac{(x_0 - \mu_0)^2}{\sigma_0^2} \right)} dx_0 \\ &= \alpha \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(\frac{\sigma_0^2 (x_1 - x_0)^2 + \sigma_x^2 (x_0 - \mu_0)^2}{\sigma_0^2 \sigma_x^2} \right)} dx_0 . \end{aligned}$$

Completing the square:

$$\begin{aligned} ax_0^2 + bx_0 + c &= \left(ax_0^2 + bx_0 + \frac{b^2}{4a} \right) + \left(c - \frac{b^2}{4a} \right) & a &= \frac{\sigma_0^2 + \sigma_x^2}{\sigma_0^2 \sigma_x^2} \\ &= a \left(x_0^2 + 2x_0 \frac{b}{2a} + \left(\frac{b}{2a} \right)^2 \right) + \left(c - \frac{b^2}{4a} \right) & b &= \frac{-2(\sigma_0^2 x_1 + \sigma_x^2 \mu_0)}{\sigma_0^2 \sigma_x^2} \\ &= a \left(x_0 + \frac{b}{2a} \right)^2 + \left(c - \frac{b^2}{4a} \right) & c &= \frac{\sigma_0^2 x_1^2 + \sigma_x^2 \mu_0^2}{\sigma_0^2 \sigma_x^2} \end{aligned}$$

$$P(x_1) = \alpha e^{-\frac{1}{2} \left(c - \frac{b^2}{4a} \right)} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left(a \left(x_0 + \frac{b}{2a} \right)^2 \right)} dx_0 = \alpha e^{-\frac{1}{2} \left(\frac{(x_1 - \mu_0)^2}{\sigma_0^2 + \sigma_x^2} \right)}$$

Hence, $P(X_1) = \mathcal{N}(\mu_0, (\sigma_0^2 + \sigma_x^2))$.

Measurement Update

$$\begin{aligned} P(x_1 | z_1) &= \alpha P(z_1 | x_1) P(x_1) \\ &= \alpha e^{-\frac{1}{2} \left(\frac{(z_1 - x_1)^2}{\sigma_z^2} \right)} e^{-\frac{1}{2} \left(\frac{(x_1 - \mu_0)^2}{\sigma_0^2 + \sigma_x^2} \right)} \\ &\quad \text{after completing the square: } P(x_1 | z_1) = \alpha e^{-\frac{1}{2} \left(\frac{(x_1 - \frac{(\sigma_0^2 + \sigma_x^2)z_1 + \sigma_z^2\mu_0}{\sigma_0^2 + \sigma_x^2 + \sigma_z^2})^2}{\frac{(\sigma_0^2 + \sigma_x^2)\sigma_z^2}{(\sigma_0^2 + \sigma_x^2 + \sigma_z^2)}} \right)} \end{aligned}$$

Hence, $P(X_{t+1} | z_{1:t+1}) = \mathcal{N}(\mu_{t+1}, \sigma_{t+1}^2)$, where

$$\mu_{t+1} = \frac{(\sigma_t^2 + \sigma_x^2)z_{t+1} + \sigma_z^2\mu_t}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2} \quad \text{and} \quad \sigma_{t+1}^2 = \frac{(\sigma_t^2 + \sigma_x^2)\sigma_z^2}{\sigma_t^2 + \sigma_x^2 + \sigma_z^2}$$

Notice that μ_{t+1} is a weighted mean between μ_t and z_{t+1} .

Numerical Example

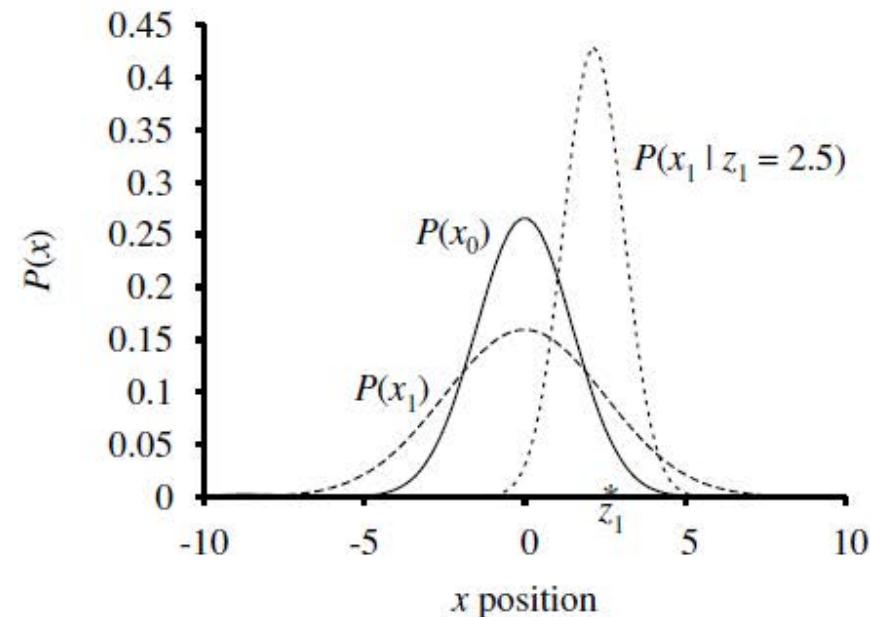


Figure 15.10 Stages in the Kalman filter update cycle for a random walk with a prior given by $\mu_0 = 0.0$ and $\sigma_0 = 1.0$, transition noise given by $\sigma_x = 2.0$, sensor noise given by $\sigma_z = 1.0$, and a first observation $z_1 = 2.5$ (marked on the x -axis). Notice how the prediction $P(x_1)$ is flattened out, relative to $P(x_0)$, by the transition noise. Notice also that the mean of the posterior distribution $P(x_1 | z_1)$ is slightly to the left of the observation z_1 because the mean is a weighted average of the prediction and the observation.

Kalman Filtering

- Multivariate Gaussian distribution

$$\mathcal{N}(\mathbf{x}; \mu, \Sigma) = \alpha \exp \left(-\frac{1}{2} (\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu) \right)$$

- Prior distribution: $P(\mathbf{X}_0) = \mathcal{N}(\mu_0, \Sigma_0)$
- Transition model: $\mathbf{X}_{t+1} = \mathbf{F}\mathbf{X}_t + \mathbf{W}_t$, where $\mathbf{W}_t \sim \mathcal{N}(\mathbf{0}, \Sigma_x)$

$$P(\mathbf{x}_{t+1} | \mathbf{x}_t) = \mathcal{N}(\mathbf{x}_{t+1}; \mathbf{F}\mathbf{x}_t, \Sigma_x)$$

- Sensor model: $\mathbf{Z}_t = \mathbf{H}\mathbf{X}_t + \mathbf{V}_t$, where $\mathbf{V}_t \sim \mathcal{N}(\mathbf{0}, \Sigma_z)$

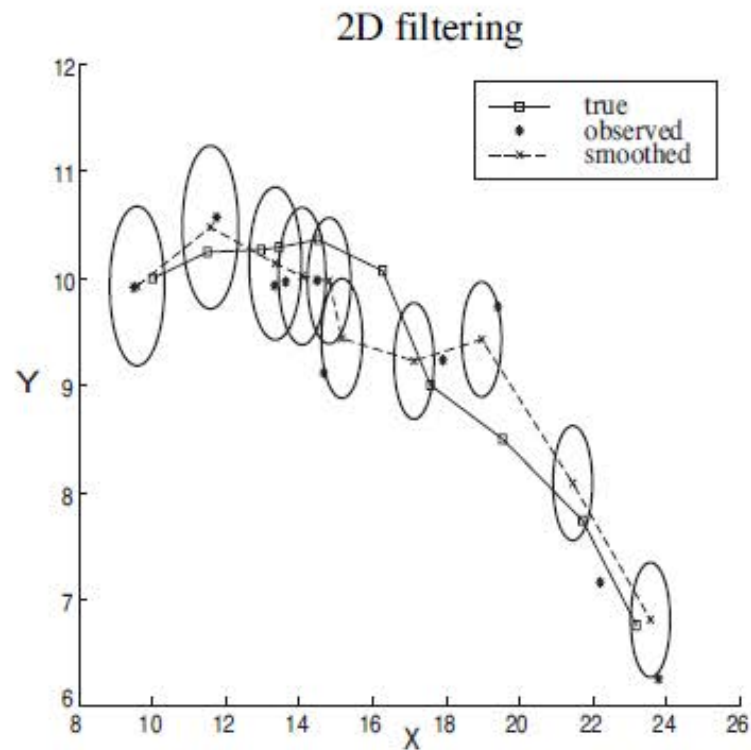
$$P(\mathbf{z}_t | \mathbf{x}_t) = \mathcal{N}(\mathbf{z}_t; \mathbf{H}\mathbf{x}_t, \Sigma_z)$$

- Update equations:

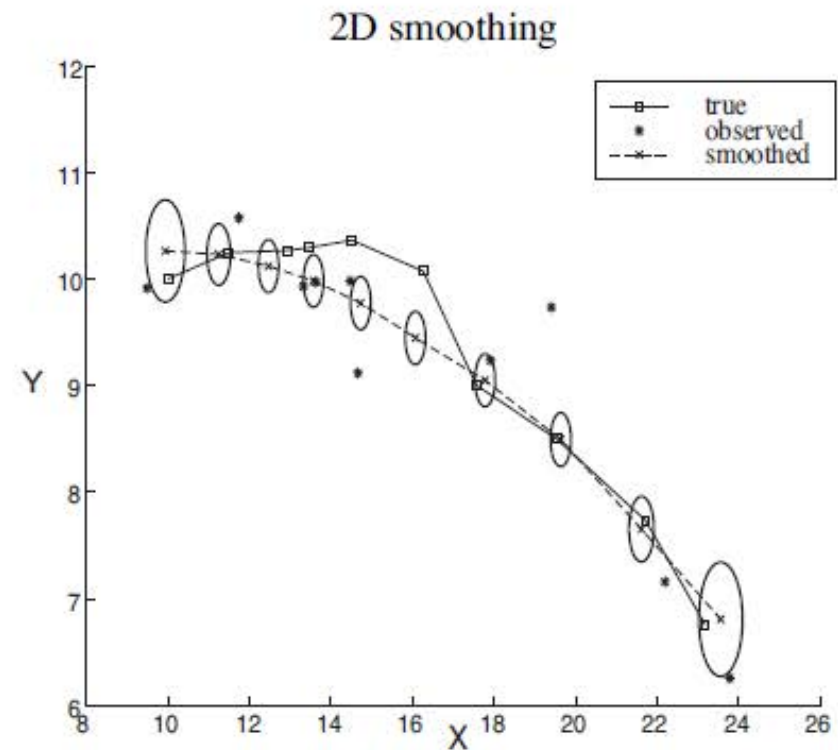
$$\begin{aligned} \mu_{t+1} &= \mathbf{F}\mu_t + \mathbf{K}_{t+1}(\mathbf{z}_{t+1} - \mathbf{H}\mathbf{F}\mu_t) \\ \Sigma_{t+1} &= (\mathbf{I} - \mathbf{K}_{t+1}\mathbf{H})(\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x) \end{aligned}$$

$$\mathbf{K}_{t+1} = (\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x)\mathbf{H}^\top (\mathbf{H}(\mathbf{F}\Sigma_t\mathbf{F}^\top + \Sigma_x)\mathbf{H}^\top + \Sigma_z)^{-1} \quad \text{Kalman Gain}$$

Numerical Example



(a)



(b)