

430.457

Introduction to Intelligent Systems

Prof. Songhwai Oh
ECE, SNU

INFERENCE IN DYNAMIC MODELS

Inference Problems in Dynamic Models

- **Filtering:** the task of computing the belief state (the posterior distribution over the most recent state) given all evidence to date, i.e., computing $P(\mathbf{X}_t | \mathbf{e}_{1:t})$.
- **Prediction:** the task of computing the posterior distribution over the future state, given all evidence to date, i.e., $P(\mathbf{X}_{t+k} | \mathbf{e}_{1:t})$ for some $k \geq 1$.
- **Smoothing:** the task of computing the posterior distribution over a past state, given all evidence up to the present, i.e., $P(\mathbf{X}_k | \mathbf{e}_{1:t})$ for some $0 \leq k \leq t$. Note that smoothing provides a better estimate of the state than filtering because it incorporates more evidence.
- **Most likely explanation:** Given a sequence of observations, find the sequence of states that is most likely to have generated those observations, i.e., $\arg \max_{\mathbf{x}_{1:t}} P(\mathbf{x}_{1:t} | \mathbf{e}_{1:t})$.
- **Learning:** Learning the transition and sensor models from observations. Note that learning requires smoothing for better estimates.

Filtering

- **Recursive estimation:** given the result of filtering up to time t , the agent needs to compute the result for $t + 1$ from the new evidence $\mathbf{e}_{t+1}$.

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) = f(\mathbf{e}_{t+1}, \mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t}))$$

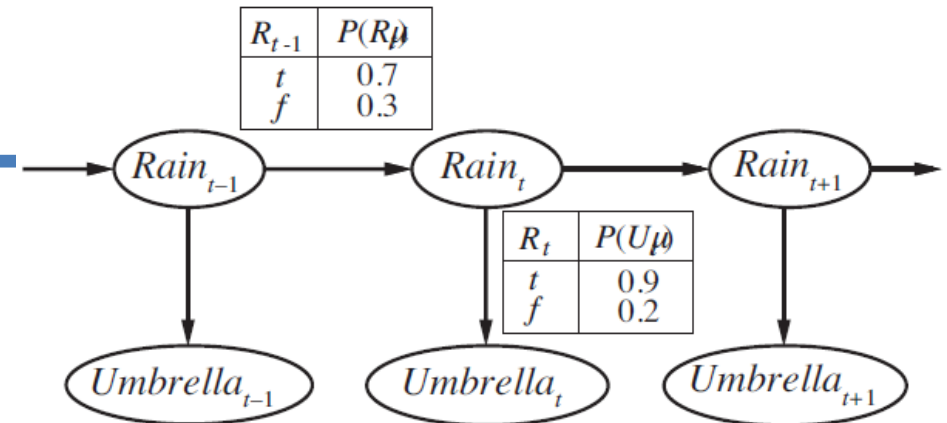
$$\begin{aligned}\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) &= \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}, \mathbf{e}_{t+1}) \quad (\text{dividing up the evidence}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}, \mathbf{e}_{1:t}) \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \quad (\text{using Bayes' rule}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \quad (\text{by the sensor Markov assumption}). \\ \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t, \mathbf{e}_{1:t}) P(\mathbf{x}_t | \mathbf{e}_{1:t}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t}) \quad (\text{Markov assumption})\end{aligned}$$

Requires two steps:

- *Prediction:* $\mathbf{P}(\mathbf{X}_t | \mathbf{e}_{1:t}) \rightarrow \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t})$
- *Measurement Update:* $\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t}) \rightarrow \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1})$

Filtering Example

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t})$$



- On day 0, we have no observations, only the security guard's prior beliefs; let's assume that consists of $\mathbf{P}(R_0) = \langle 0.5, 0.5 \rangle$.
- On day 1, the umbrella appears, so $U_1 = \text{true}$. The prediction from $t = 0$ to $t = 1$ is

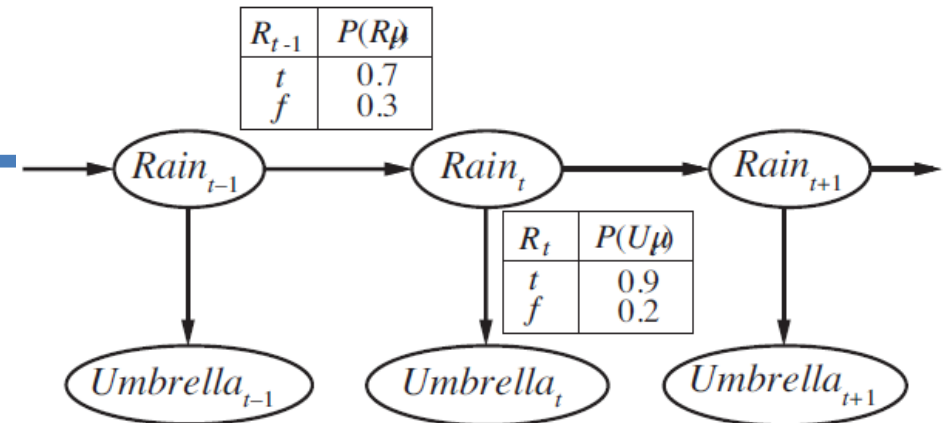
$$\begin{aligned} \mathbf{P}(R_1) &= \sum_{r_0} \mathbf{P}(R_1 | r_0) P(r_0) \\ &= \langle 0.7, 0.3 \rangle \times 0.5 + \langle 0.3, 0.7 \rangle \times 0.5 = \langle 0.5, 0.5 \rangle . \end{aligned}$$

Then the update step simply multiplies by the probability of the evidence for $t = 1$ and normalizes, as shown in Equation (15.4):

$$\begin{aligned} \mathbf{P}(R_1 | u_1) &= \alpha \mathbf{P}(u_1 | R_1) \mathbf{P}(R_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.5, 0.5 \rangle \\ &= \alpha \langle 0.45, 0.1 \rangle \approx \langle 0.818, 0.182 \rangle . \end{aligned}$$

Filtering Example

$$\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) = \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \sum_{\mathbf{x}_t} \mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) P(\mathbf{x}_t | \mathbf{e}_{1:t})$$



$$\mathbf{P}(R_1 | u_1) = \langle 0.818, 0.182 \rangle$$

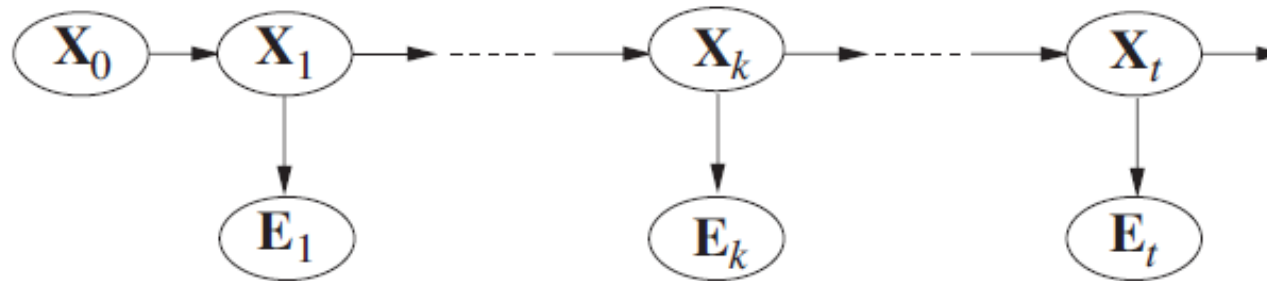
- On day 2, the umbrella appears, so $U_2 = \text{true}$. The prediction from $t = 1$ to $t = 2$ is

$$\begin{aligned} \mathbf{P}(R_2 | u_1) &= \sum_{r_1} \mathbf{P}(R_2 | r_1) P(r_1 | u_1) \\ &= \langle 0.7, 0.3 \rangle \times 0.818 + \langle 0.3, 0.7 \rangle \times 0.182 \approx \langle 0.627, 0.373 \rangle, \end{aligned}$$

and updating it with the evidence for $t = 2$ gives

$$\begin{aligned} \mathbf{P}(R_2 | u_1, u_2) &= \alpha \mathbf{P}(u_2 | R_2) \mathbf{P}(R_2 | u_1) = \alpha \langle 0.9, 0.2 \rangle \langle 0.627, 0.373 \rangle \\ &= \alpha \langle 0.565, 0.075 \rangle \approx \langle 0.883, 0.117 \rangle. \end{aligned}$$

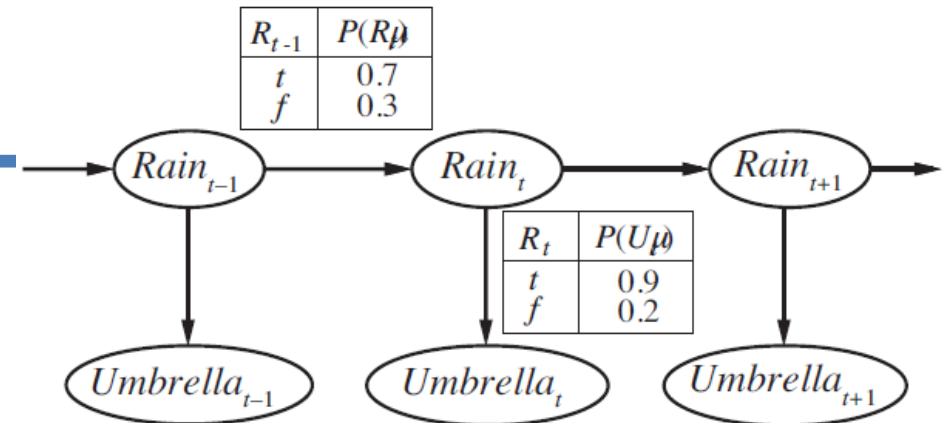
Smoothing



$$\begin{aligned}\mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:t}) &= \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}, \mathbf{e}_{k+1:t}) \\ &= \alpha \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k, \mathbf{e}_{1:k}) \quad (\text{using Bayes' rule}) \\ &= \alpha \mathbf{P}(\mathbf{X}_k \mid \mathbf{e}_{1:k}) \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) \quad (\text{using conditional independence}) \\ &= \alpha \mathbf{f}_{1:k} \times \mathbf{b}_{k+1:t} .\end{aligned}$$

$$\begin{aligned}\mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k) &= \sum_{\mathbf{x}_{k+1}} \mathbf{P}(\mathbf{e}_{k+1:t} \mid \mathbf{X}_k, \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \quad (\text{conditioning on } \mathbf{X}_{k+1}) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \quad (\text{by conditional independence}) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1}, \mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} \mid \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} \mid \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} \mid \mathbf{X}_k)\end{aligned}$$

Smoothing Example



$$\mathbf{P}(R_1 | u_1, u_2) = \alpha \mathbf{P}(R_1 | u_1) \mathbf{P}(u_2 | R_1)$$

$$\mathbf{P}(R_1 | u_1) = \langle 0.818, 0.182 \rangle$$

$$\begin{aligned} \mathbf{P}(u_2 | R_1) &= \sum_{r_2} P(u_2 | r_2) P(r_2 | R_1) \\ &= (0.9 \times 1 \times \langle 0.7, 0.3 \rangle) + (0.2 \times 1 \times \langle 0.3, 0.7 \rangle) = \langle 0.69, 0.41 \rangle \end{aligned}$$

$$\mathbf{P}(R_1 | u_1, u_2) = \alpha \langle 0.818, 0.182 \rangle \times \langle 0.69, 0.41 \rangle \approx \langle 0.883, 0.117 \rangle$$

$$\sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) \mathbf{P}(\mathbf{x}_{k+1} | \mathbf{X}_k)$$

Forward and Backward Recursions

- Forward recursion

$$\begin{aligned}\mathbf{f}_{1:k+1} &:= P(\mathbf{X}_{k+1} | \mathbf{e}_{1:k+1}) \\ &= \alpha P(\mathbf{e}_{k+1} | \mathbf{X}_{k+1}) \sum_{\mathbf{x}_k} P(\mathbf{X}_{k+1} | \mathbf{x}_k) P(\mathbf{x}_k | \mathbf{e}_{1:k}) \\ &= \alpha \cdot \textit{Forward}(\mathbf{f}_{1:k}, \mathbf{e}_{k+1})\end{aligned}$$

Initialized with $\mathbf{f}_{1:0} = P(\mathbf{X}_0)$.

- Backward recursion

$$\begin{aligned}\mathbf{b}_{k+1:t} &:= P(\mathbf{e}_{k+1:t} | \mathbf{X}_k) \\ &= \sum_{\mathbf{x}_{k+1}} P(\mathbf{e}_{k+1} | \mathbf{x}_{k+1}) P(\mathbf{e}_{k+2:t} | \mathbf{x}_{k+1}) P(\mathbf{x}_{k+1} | \mathbf{X}_k) \\ &= \textit{Backward}(\mathbf{b}_{k+2:t}, \mathbf{e}_{k+1})\end{aligned}$$

Initialized with $\mathbf{b}_{t+1:t} = \mathbf{1}$.

Forward-Backward Algorithm

```
function FORWARD-BACKWARD(ev, prior) returns a vector of probability distributions
  inputs: ev, a vector of evidence values for steps 1, ..., t
           prior, the prior distribution on the initial state,  $P(X_0)$ 
  local variables: fv, a vector of forward messages for steps 0, ..., t
                    b, a representation of the backward message, initially all 1s
                    sv, a vector of smoothed estimates for steps 1, ..., t

  fv[0]  $\leftarrow$  prior
  for i = 1 to t do
    fv[i]  $\leftarrow$  FORWARD(fv[i - 1], ev[i])
  for i = t downto 1 do
    sv[i]  $\leftarrow$  NORMALIZE(fv[i]  $\times$  b)
    b  $\leftarrow$  BACKWARD(b, ev[i])
  return sv
```

Finding the Most Likely Sequence

- Consider the umbrella problem and we have observations for 5 days
 - Umbrella_1 = true
 - Umbrella_2 = true
 - Umbrella_3 = false
 - Umbrella_4 = true
 - Umbrella_5 = true
- What is the most likely weather sequence?
 - [Rain, Rain, Rain, Rain, Rain]?
 - [Rain, Rain, No Rain, Rain, Rain]?
 - [Rain, Rain, No Rain, No Rain, Rain]?
 - There are 2^5 possible sequences.

Viterbi Algorithm

- Viterbi algorithm is used to find the most likely sequence.
- There is a recursive relationship between most likely paths to each state $\mathbf{x}_{t+1}$ and most likely paths to each state $\mathbf{x}_t$. (**Principle of Optimality**)

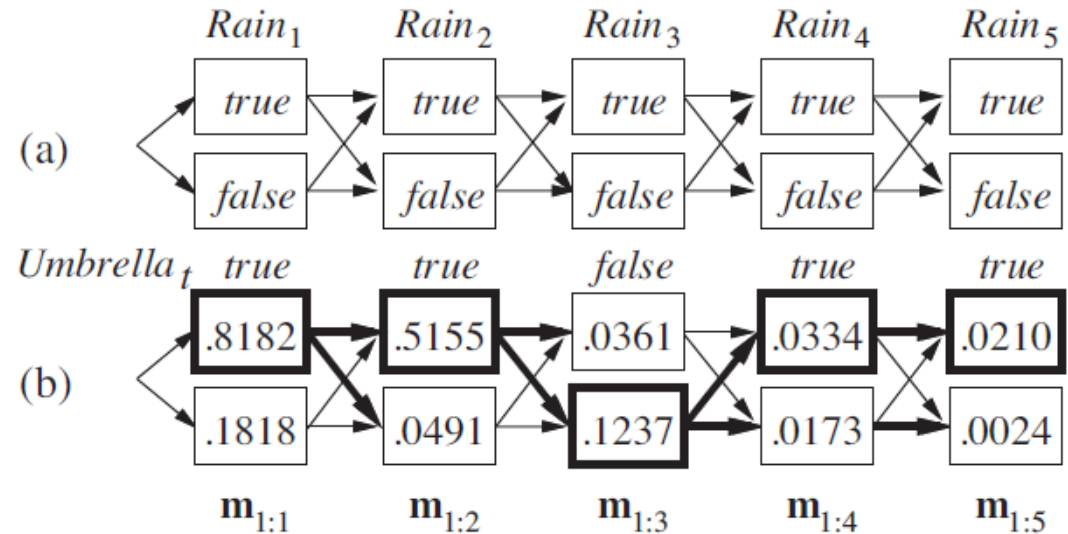
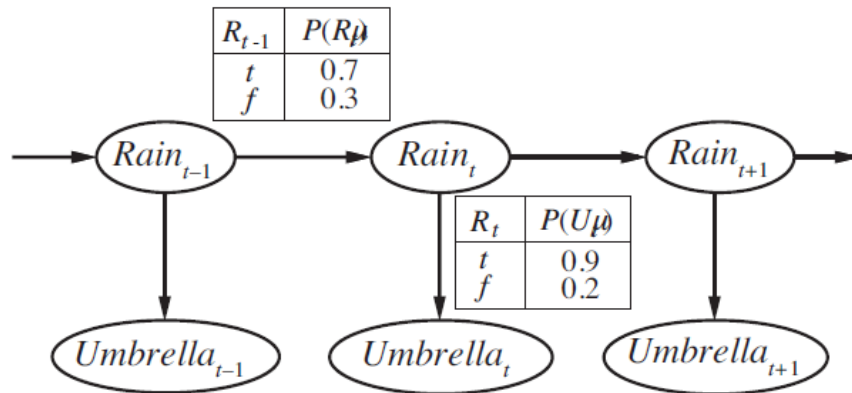
$$\begin{aligned} & \max_{\mathbf{x}_1 \dots \mathbf{x}_t} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_t, \mathbf{X}_{t+1} | \mathbf{e}_{1:t+1}) \\ &= \alpha \mathbf{P}(\mathbf{e}_{t+1} | \mathbf{X}_{t+1}) \max_{\mathbf{x}_t} \left(\mathbf{P}(\mathbf{X}_{t+1} | \mathbf{x}_t) \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} P(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{x}_t | \mathbf{e}_{1:t}) \right) \end{aligned}$$

$$\mathbf{m}_{1:t} = \max_{\mathbf{x}_1 \dots \mathbf{x}_{t-1}} \mathbf{P}(\mathbf{x}_1, \dots, \mathbf{x}_{t-1}, \mathbf{X}_t | \mathbf{e}_{1:t})$$

cf. Forward recursion

$$\begin{aligned} \mathbf{f}_{1:k+1} &:= P(\mathbf{X}_{k+1} | \mathbf{e}_{1:k+1}) \\ &= \alpha P(\mathbf{e}_{k+1} | \mathbf{X}_{k+1}) \sum_{\mathbf{x}_k} P(\mathbf{X}_{k+1} | \mathbf{x}_k) P(\mathbf{x}_k | \mathbf{e}_{1:k}) \\ &= \alpha \cdot \textit{Forward}(\mathbf{f}_{1:k}, \mathbf{e}_{k+1}) \end{aligned}$$

Viterbi Algorithm: Example

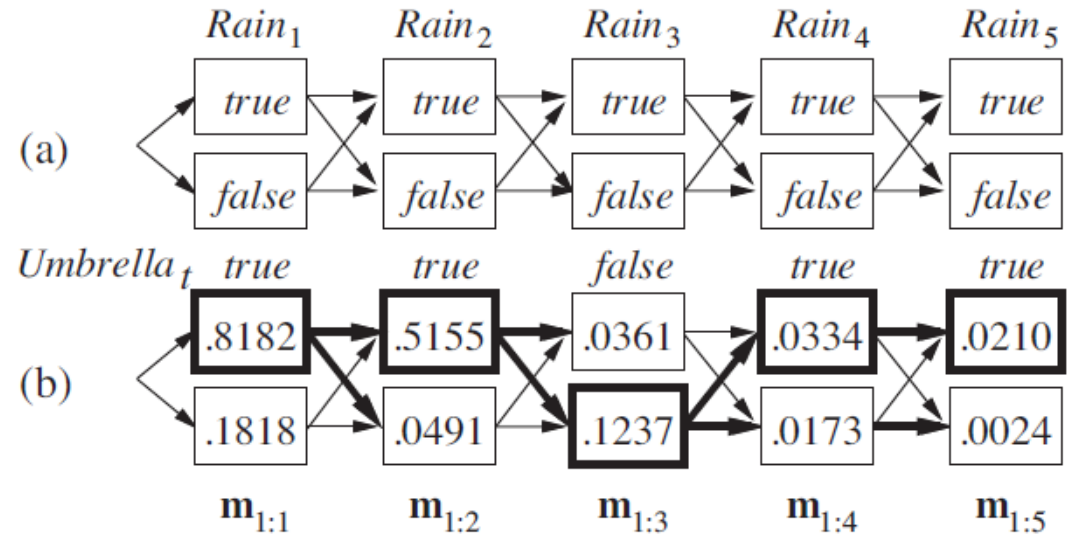
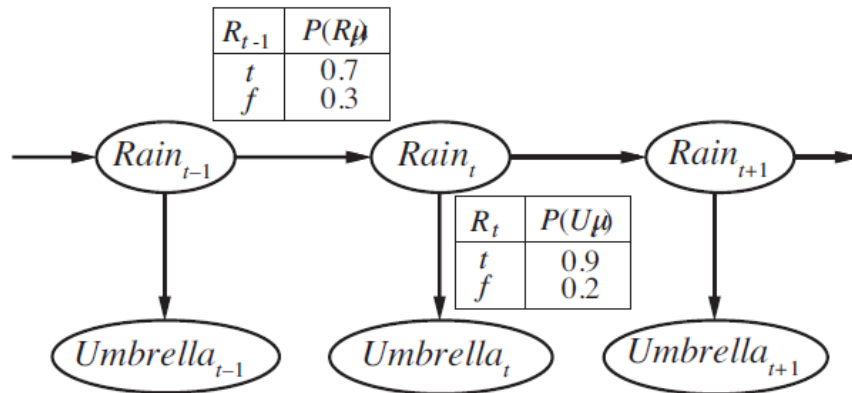


$$\mathbf{m}_{1:1} = P(\mathbf{X}_1 | \mathbf{e}_{1:1}) = P(R_1 | u_1 = t) = \langle 0.8182, 0.1818 \rangle$$

$$\begin{aligned} \mathbf{m}_{1:2} &= \alpha P(u_2 = t | R_2) \max_{r_1} (P(R_2 | r_1) \mathbf{m}_{1:1}(r_1)) \\ &= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.7, 0.3 \rangle 0.8182, \langle 0.3, 0.7 \rangle 0.1818) \\ &= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.5727, 0.2455 \rangle, \langle 0.0546, 0.1274 \rangle) \\ &= \alpha \langle 0.9, 0.2 \rangle \langle 0.5727, 0.2455 \rangle \\ &= \alpha \langle 0.5155, 0.0491 \rangle \end{aligned}$$

$$\arg \max(\cdot) = \langle 1, 1 \rangle$$

Viterbi Algorithm: Example



$$\mathbf{m}_{1:2} = \alpha \langle 0.5155, 0.0491 \rangle$$

$$\mathbf{m}_{1:3} = \alpha P(u_3 = f | R_3) \max_{r_2} (P(R_3 | r_2) \mathbf{m}_{1:2}(r_2))$$

$$= \alpha \langle 0.1, 0.8 \rangle \max (\langle 0.7, 0.3 \rangle 0.5155, \langle 0.3, 0.7 \rangle 0.0491)$$

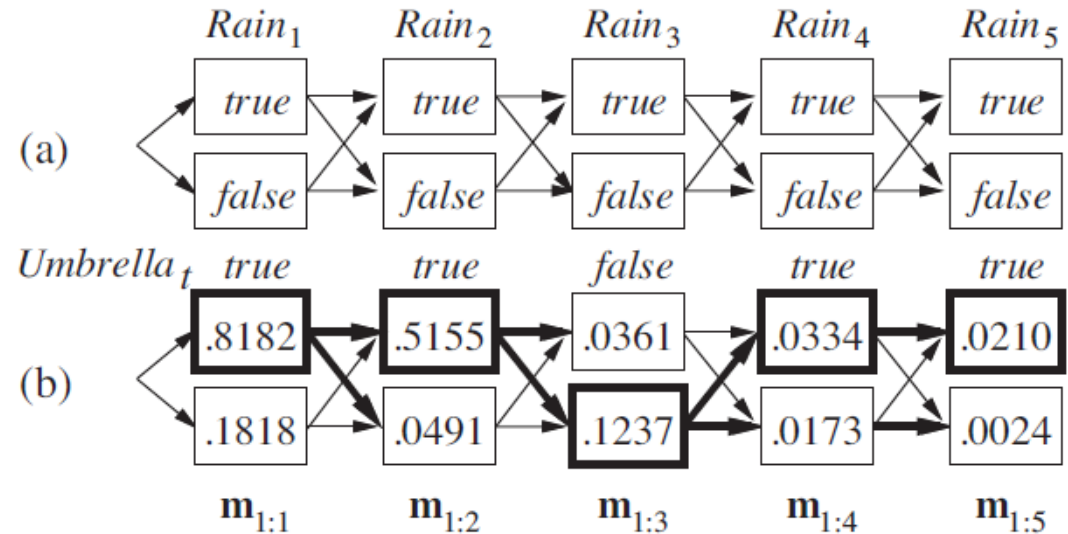
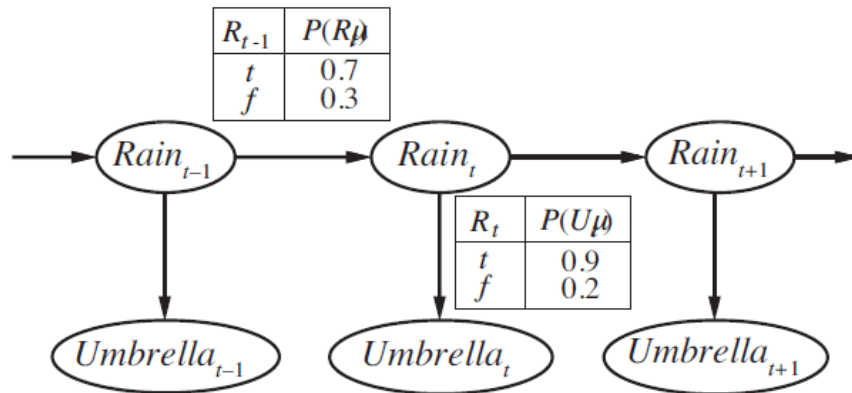
$$= \alpha \langle 0.1, 0.8 \rangle \max (\langle 0.3608, 0.1546 \rangle, \langle 0.0147, 0.0344 \rangle)$$

$$= \alpha \langle 0.1, 0.8 \rangle \langle 0.3608, 0.1546 \rangle$$

$$= \alpha \langle 0.0361, 0.1237 \rangle$$

$$\arg \max(\cdot) = \langle 1, 1 \rangle$$

Viterbi Algorithm: Example



$$\mathbf{m}_{1:3} = \alpha \langle 0.0361, 0.1237 \rangle$$

$$\mathbf{m}_{1:4} = \alpha P(u_4 = t | R_4) \max_{r_3} (P(R_4 | r_3) \mathbf{m}_{1:3}(r_3))$$

$$= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.7, 0.3 \rangle 0.0361, \langle 0.3, 0.7 \rangle 0.1237)$$

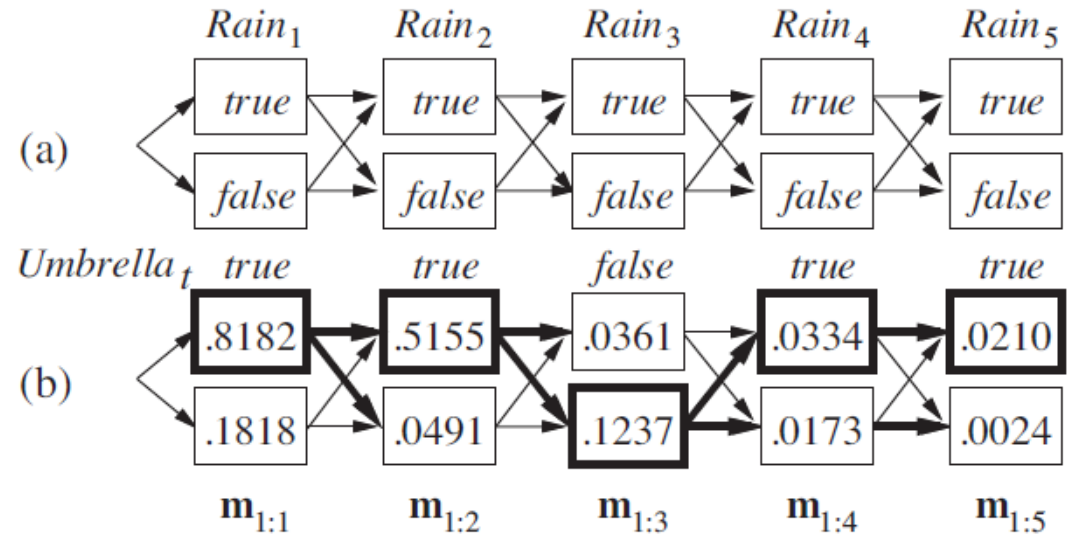
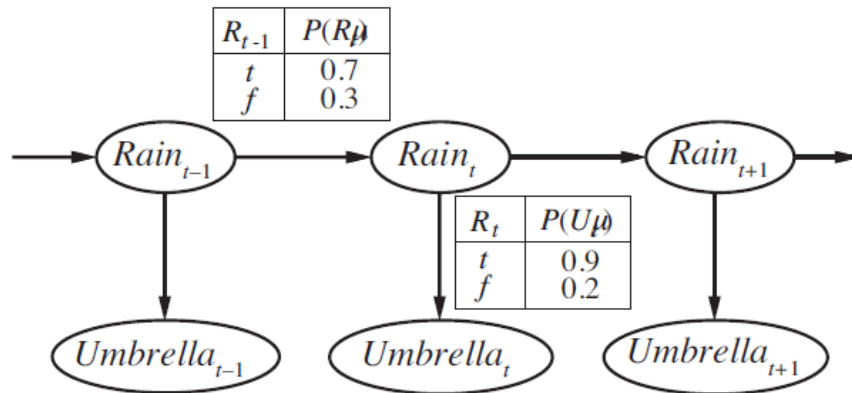
$$= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.0252, 0.0108 \rangle, \langle 0.0371, 0.0866 \rangle)$$

$$= \alpha \langle 0.9, 0.2 \rangle \langle 0.0371, 0.0866 \rangle$$

$$= \alpha \langle 0.0334, 0.0173 \rangle$$

$$\arg \max(\cdot) = \langle 2, 2 \rangle$$

Viterbi Algorithm: Example



$$\mathbf{m}_{1:4} = \alpha \langle 0.0334, 0.0173 \rangle$$

$$\mathbf{m}_{1:5} = \alpha P(u_5 = t | R_5) \max_{r_4} (P(R_5 | r_4) \mathbf{m}_{1:4}(r_4))$$

$$= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.7, 0.3 \rangle 0.0334, \langle 0.3, 0.7 \rangle 0.0173)$$

$$= \alpha \langle 0.9, 0.2 \rangle \max (\langle 0.0234, 0.0100 \rangle, \langle 0.0052, 0.0121 \rangle)$$

$$= \alpha \langle 0.9, 0.2 \rangle \langle 0.0234, 0.0121 \rangle$$

$$= \alpha \langle 0.0210, 0.0024 \rangle$$

$$\arg \max(\cdot) = \langle 1, 2 \rangle$$