

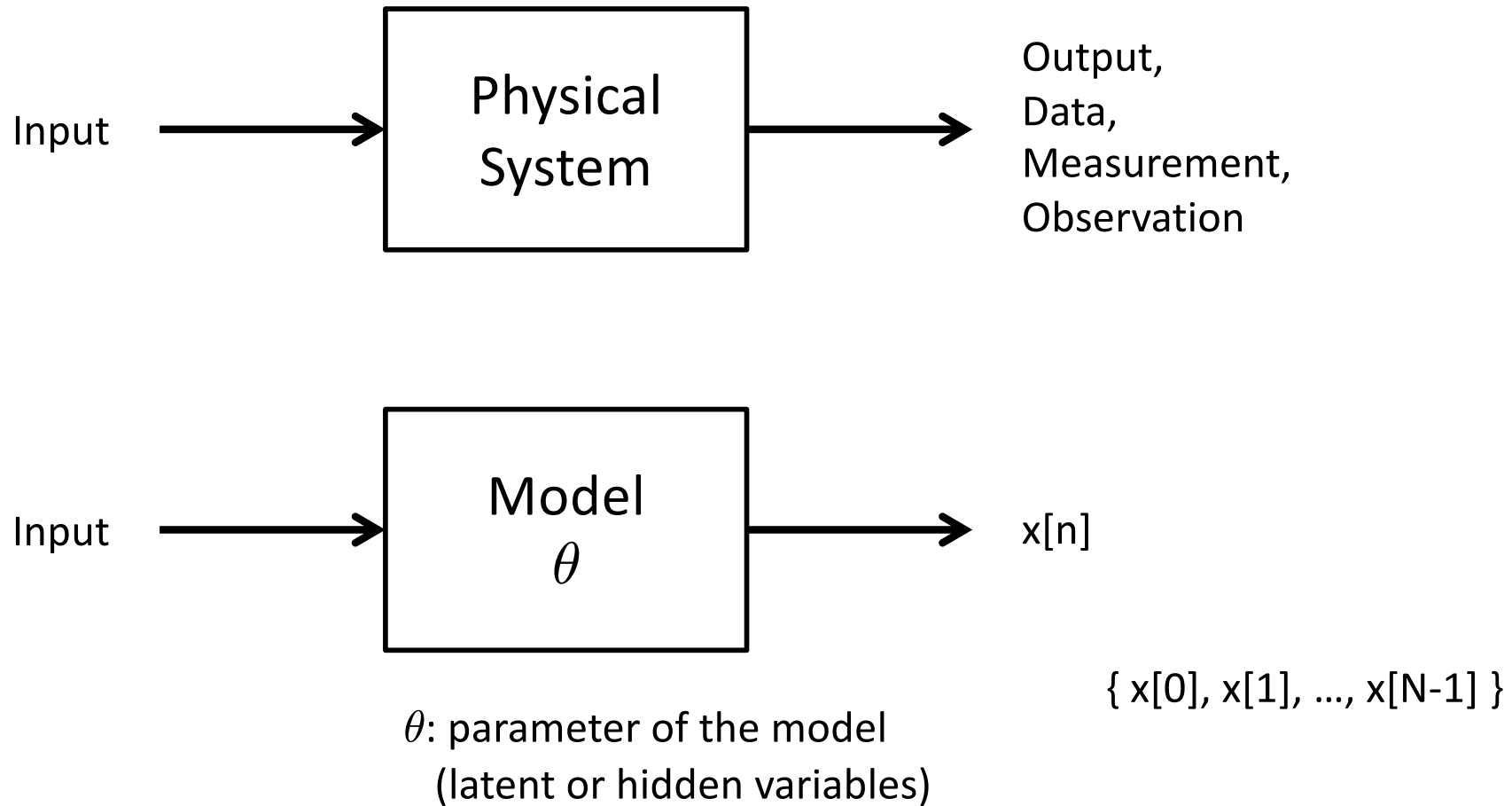
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Estimation Theory

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MINIMUM VARIANCE UNBIASED ESTIMATION

Problem Formulation



Estimation Problem

Estimation problem (parameter estimation) is to determine θ based on the measurement data.

Estimator: $\hat{\theta} := g(x[0], x[1], \dots, x[N-1])$



A function of all available data

→ $\hat{\theta}$ is a random variable.

Issues:

- How to choose an estimator?
 - What criteria to use?
- What is the best possible performance?

Bias

Bias of an estimator $\hat{\theta}$:

$$b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$$

$\hat{\theta}$ is an *unbiased* estimator of θ if $\mathbb{E}(\hat{\theta}) = \theta$.

$\hat{\theta}$ is a *biased* estimator of θ if $\mathbb{E}(\hat{\theta}) \neq \theta$.

Example: $x[n] = A + w[n]$ $w[n] \sim \mathcal{N}(0, \sigma_n^2)$ for $n = 0, 1, \dots, N-1$

- *unbiased*: $\hat{A} = \frac{1}{N} \sum_{n=0}^{N-1} x[n]$

- *biased*: $\check{A} = \frac{1}{2N} \sum_{n=0}^{N-1} x[n]$

$$\mathbb{E}(\check{A}) = \frac{1}{2}A \quad \begin{array}{ll} = A & \text{if } A = 0 \\ \neq A & \text{if } A \neq 0. \end{array}$$

Bias: $b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$

- Let $\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n$ be multiple estimates of θ .
- Let $\hat{\theta} = \frac{1}{n} \sum_{i=1}^n \hat{\theta}_i$.
- If unbiased and uncorrelated with the same variance,

$$\begin{aligned}\mathbb{E}(\hat{\theta}) &= \theta \\ \text{var}(\hat{\theta}) &= \frac{1}{n^2} \sum_{i=1}^n \text{var}(\hat{\theta}_i) = \frac{1}{n} \text{var}(\theta_1).\end{aligned}$$

Hence, as $n \rightarrow \infty$, $\text{var}(\hat{\theta}) \rightarrow 0$ and $\hat{\theta} \rightarrow \theta$.

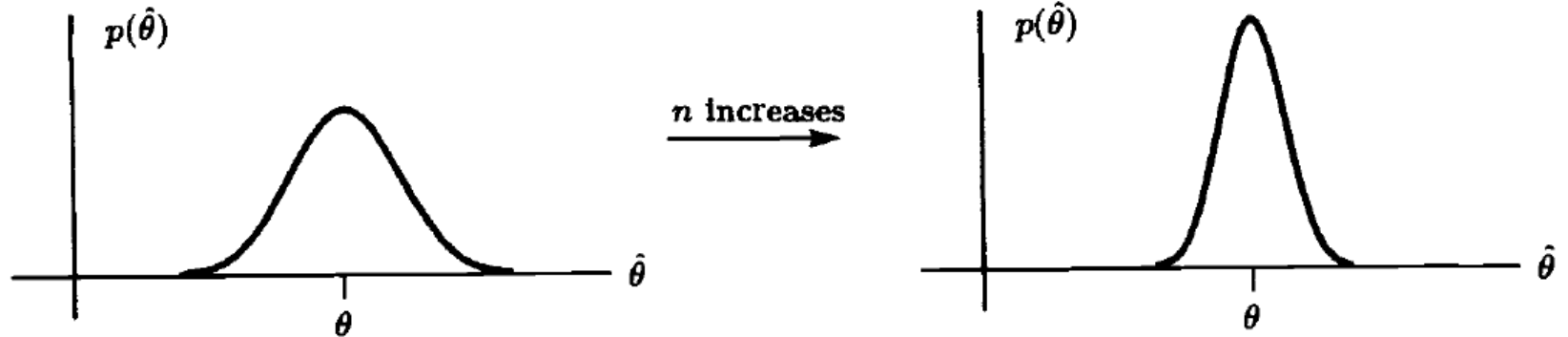
- If biased and $\mathbb{E}(\hat{\theta}_i) = \theta + b(\theta)$,

$$\mathbb{E}(\hat{\theta}) = \frac{1}{n} \sum_{i=1}^n \mathbb{E}(\hat{\theta}_i) = \theta + b(\theta).$$

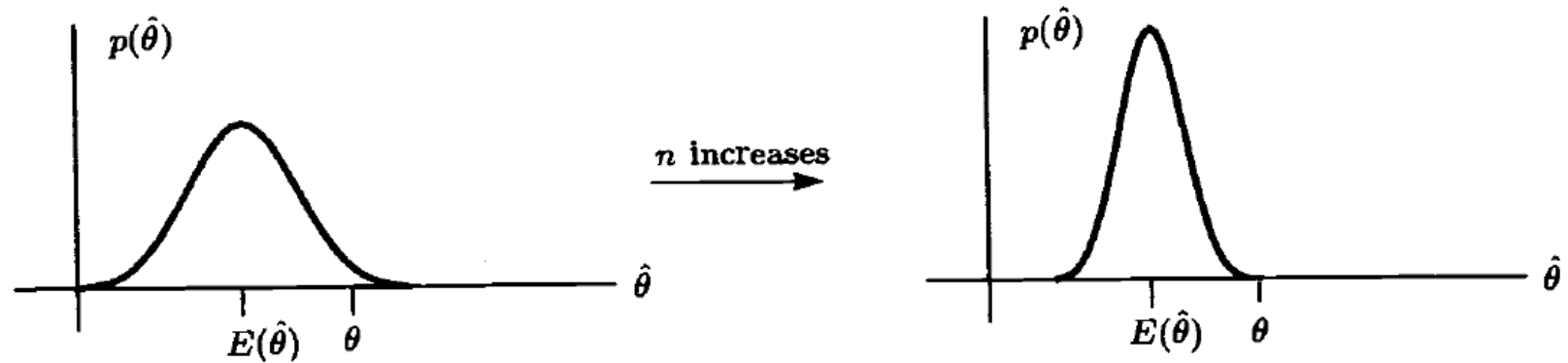
Hence, as $n \rightarrow \infty$, $\hat{\theta} \not\rightarrow \theta$.

Bias: $b(\theta) = \mathbb{E}(\hat{\theta}) - \theta$

Unbiased estimator



Biased estimator



Mean Square Error (MSE)

$$\text{mse}(\hat{\theta}) = \mathbb{E} \left(\hat{\theta} - \theta \right)^2$$

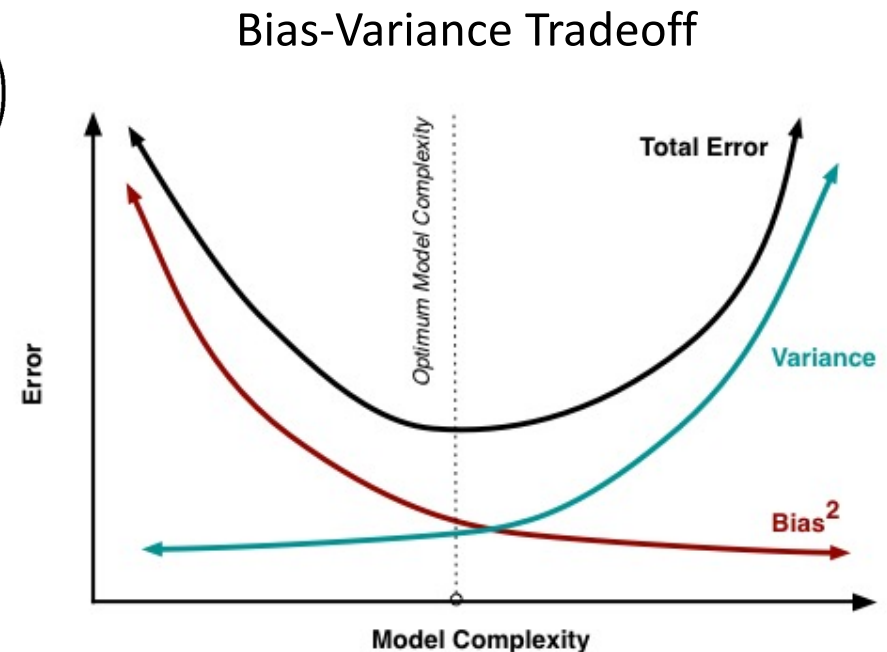
Important: θ is not a random variable.

$$\begin{aligned} \text{mse}(\hat{\theta}) &= \mathbb{E} \left(\hat{\theta} - \theta \right)^2 \\ &= \mathbb{E} \left[\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) + \mathbb{E}(\hat{\theta}) - \theta \right)^2 \right] \\ &= \mathbb{E} \left[\left(\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) \right) + \left(\mathbb{E}(\hat{\theta}) - \theta \right) \right)^2 \right] \\ &= \mathbb{E} \left(\left(\hat{\theta} - \mathbb{E}(\hat{\theta}) \right)^2 \right) + \mathbb{E} \left(\left(\mathbb{E}(\hat{\theta}) - \theta \right)^2 \right) \\ &= \text{var}(\hat{\theta}) + \left(\mathbb{E}(\hat{\theta}) - \theta \right)^2 \\ &= \text{var}(\hat{\theta}) + b(\theta)^2 \end{aligned}$$

↖ A function of θ .

We cannot compute $\text{mse}(\hat{\theta})$ without knowing θ in general.

Hence, not a good criterion to use.



Minimum Variance Unbiased Estimator

- Find an estimator which minimizes the variance from a set of unbiased estimators.

Cramer-Rao Lower Bound (CRLB)

- Lower bound of the variance of an unbiased estimator

Rao-Blackwell-Lehmann-Scheffe (RBLS) Theorem

- Sufficient statistics $\rightarrow$ an unbiased estimator