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Estimation Theory

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REVIEW OF LINEAR SYSTEM THEORY

Matrix

$$
\begin{aligned}
(\mathbf{AB})^T &= \mathbf{B}^T\mathbf{A}^T \\
\mathbf{A}\mathbf{A}^{-1} &= \mathbf{A}^{-1}\mathbf{A} = \mathbf{I} \\
(\mathbf{AB})^{-1} &= \mathbf{B}^{-1}\mathbf{A}^{-1} \\
\left(\mathbf{A}^T\right)^{-1} &= \left(\mathbf{A}^{-1}\right)^T
\end{aligned}
\qquad
\begin{aligned}
\mathrm{Tr}(\mathbf{AB}) &= \mathrm{Tr}(\mathbf{BA}) \\
\mathrm{Tr}(\mathbf{ABC}) &= \mathrm{Tr}(\mathbf{CAB}) = \mathrm{Tr}(\mathbf{BCA}) \\
|\mathbf{AB}| &= |\mathbf{A}||\mathbf{B}| \\
|\mathbf{A}^{-1}| &= \frac{1}{|\mathbf{A}|}
\end{aligned}
$$

For $\mathbf{A} \in \mathbb{R}^{n \times n}$, the eigenvector equation is defined as $\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \qquad \text{for } i = 1, \ldots, n.$	$\lvert\mathbf{A} - \lambda\mathbf{I}\rvert = 0$ $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$

- $\mathbf{A}$ is *positive definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$.
- $\mathbf{A}$ is *positive semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.
- $\mathbf{A}$ is *negative definite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ for all nonzero $\mathbf{x} \in \mathbb{R}^n$.
- $\mathbf{A}$ is *negative semidefinite* if $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$.

Linear System

$$\boxed{\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}}$$

$x \in \mathbb{R}^n,\ A \in \mathbb{R}^{n \times n}$

Solution:

$$\begin{aligned} x(t) &= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau \\ y(t) &= Cx(t) \end{aligned}$$

State transition matrix: $e^{At} = \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$

Stability

$$\begin{aligned} \dot{x} &= Ax \\ y &= Cx \end{aligned}$$

- *Marginally stable* if $x(t)$ is bounded for all t and for all bounded initial state $x(0)$.
- *Asymptotically stable* if, for all bounded initial state $x(0)$, $\lim_{t\to\infty} x(t) = 0$.

Theorems

- A continuous-time linear time invariant (LTI) system is marginally stable if and only if $\lim_{t\to\infty} \exp(At) \leq M < \infty$ for some matrix M.
- A continuous-time LTI system is asymptotically stable if and only if $\lim_{t\to\infty} \exp(At) = 0$.
- A continuous-time LTI system is marginally stable if and only if all of the eigenvalues of A have negative real parts or all of the eigenvalues of A have negative or zero real parts, and those with zero real parts have a geometric multiplicity equal to their algebraic multiplicity.
- A continuous-time LTI system is asymptotically stable if and only if all of the eigenvalues of A have negative real parts.

Controllability

$$\boxed{\dot{x} = Ax + Bu}$$

(A, B) is said to be *controllable* if for any initial state $x(0)$ and any final state x_1, there exists an input that transfers $x(0)$ to x_1 in a finite time.

Theorem: The following statements are equivalent.

- (A, B) is controllable.
- The controllability matrix $[B \; AB \; \cdots \; A^{n-1}B]$ has full row rank.
- The controllability Gramian

  $$\int_0^t e^{A\tau} B B^T e^{A^T \tau} d\tau$$

  is nonsingular for any $t > 0$.
- $[A - \lambda I \;\; B]$ has full row rank at every eigenvalue of A.
- If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

  $$A W_c + W_c A^T = -B B^T$$

  is positive definite.

Observability

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

(A, C) is said to be *observable* if for any initial state $x(0)$ and any final time $t > 0$ the initial state $x(0)$ can be uniquely determined by knowledge of the input $u(s)$ and output $y(s)$ for all $s \in [0, t]$.

Theorem: The following statements are equivalent.

- (A, C) is observable.
- The observability matrix

$$\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

  has full column rank.

- The observability Gramian

$$\int_0^t e^{A^T\tau} C^T C e^{A\tau} d\tau$$

is nonsingular for any $t > 0$.

- $\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$ has full column rank at every eigenvalue of A.

- If, in addition, all eigenvalues of A have negative real parts, then the unique solution of

$$A^T W_o + W_o A = -C^T C$$

is positive definite.

Stabilizability and Detectability

- If a system is controllable or stable, then it is also *stabilizable*. If a system is uncontrollable or unstable, then it is *stabilizable* if its uncontrollable modes are stable.
- If a system is observable or stable, then it is also *detectable*. If a system is unobservable or unstable, then it is *detectable* if its unobservable modes are stable.