

430.714
Estimation Theory

Midterm

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Student Name:

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IMPORTANT:

- Read all problems carefully and start with easier ones.
- You need to show all your steps for full credit.

Problem 1: [30 pts]

Consider $\mathbf{x} = [x_1, x_2, \dots, x_n]^T$, where x_i is independent and identically distributed with density $p(x; \theta)$. Suppose that $p(x; \theta)$ satisfies the regularity condition. Let

$$\mathcal{I}_1(\theta) = \mathbb{E} \left[\left(\frac{\partial}{\partial \theta} \log p(x_1; \theta) \right)^2 \right].$$

- (a) Find $\mathcal{I}_1(\theta)$ in terms of Fisher information $\mathcal{I}(\theta) = \mathbb{E} \left[\left(\frac{\partial}{\partial \theta} \log p(\mathbf{x}; \theta) \right)^2 \right]$. (Show all your steps for full credit.)
- (b) Suppose that $p(x; \theta) = \mathcal{N}(\theta, \sigma^2)$. Compute $\mathcal{I}_1(\theta)$.
- (c) Suppose that $p(x; \theta) = \mathcal{N}(\theta, \sigma^2)$. Using results from (a) and (b), show that $\bar{x} = \frac{1}{n} \sum_{i=1}^n x_i$ is the minimum variance unbiased estimator of θ .

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Problem 2: [40 pts]

Consider the following measurement model

$$x[n] = \sum_{k=0}^K a_k n^k + w[n],$$

for $n = 0, \dots, N-1$. Let $\mathbf{x} = [x[0], x[1], \dots, x[N-1]]^T$, $\mathbf{a} = [a_0, a_1, \dots, a_K]^T$, and $\mathbf{w} = [w[0], w[1], \dots, w[N-1]]^T$. Suppose that $\mathbf{w} \sim \mathcal{N}(\mathbf{0}, \mathbf{\Sigma})$.

- (a) Show that $p(\mathbf{x}; \mathbf{a})$ satisfies the regularity condition.
- (b) Find the MVU estimator $\hat{\mathbf{a}}_{\text{MVU}}$ for $\mathbf{a}$.
- (c) Show that $\hat{\mathbf{a}}_{\text{MVU}}$ is an efficient estimator.
- (d) Compute the covariance matrix of $\hat{\mathbf{a}}_{\text{MVU}}$.
- (e) What is the best linear unbiased estimator $\hat{\mathbf{a}}_{\text{BLUE}}$ for $\mathbf{a}$?
- (f) Let $\mathbf{W}$ be a weighting matrix. Write down the weighted least squares cost function and find the least squares estimator $\hat{\mathbf{a}}_{\text{WLS}}$ for $\mathbf{a}$.

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Problem 3: [30 pts]

Consider the following system

$$y = a_1x_1 + a_2x_2 + w,$$

where x_1 , x_2 , and w are independent zero mean Gaussian random variables with variance σ_1^2 , σ_2^2 , and σ_w^2 , respectively.

- (a) Find the joint distribution of x_1 , x_2 , and y .
- (b) Find the conditional distribution of x_1 and x_2 given y .
- (c) Find the MMSE estimate of x_1 and x_2 from the measurement y .
- (d) Find the MAP estimate of $\mathbf{x} = [x_1 \ x_2]^T$ given y by taking the derivative of the posterior.
- (e) Discuss the relationship between the MMSE and MAP estimators you have found above.

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